UDC 62-50

ON THE THEORY OF DIFFERENTIAL GAMES IN SYSTEMS WITH AFTEREFFECT

PMM Vol. 42, No. 6, 1978 pp. 969-977

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(Received January 20, 1978)

An encounter-evasion differential game is studied for control systems with aftereffect [1 —4]. A feature of the system being analyzed is that it has a time-lag effect with respect to the controls which provides the system with important new peculiarities. Using the investigations in [1 —4], conditions for the solvability of the problem are indicated and the required control procedures are constructed.

1. The control system

$$x^{\cdot} = f_1(t, x, u, u^\tau) + f_2(t, x, v) \tag{1.1}$$

$$u^\tau = u(t-\tau), \quad t \in [t_0, \vartheta], \quad \tau = \text{const}, \quad 0 < \tau < \vartheta - t_0$$

$$\| f_1(t, x, u, u^\tau) + f_2(t, x, v) \| \leqslant \varkappa(1 + \|x\|), \quad \varkappa = \text{const}$$

is given. Here x is the n -dimensional phase vector; the r_1 -dimensional vector u and the r_2 -dimensional vector v are controls subject to the conditions $u \in P$ and $v \in Q$, where P and Q are compacta; the r_1 -dimensional vector u^τ is connected to vector u by the relation shown; the functions $f_1(t, x, u, u^\tau)$ and $f_2(t, x, v)$ are defined, continuous and continuously differentiable in x on $[t_0, \vartheta] \times E_n \times P \times P$ and $[t_0, \vartheta] \times E_n \times Q$ (E_n is the n - dimensional Euclidean space), respectively and the condition stated is fulfilled in the domain of definition.

The encounter problem consists in choosing the control u that takes the phase vector of system (1.1) onto a specified set M in specified time, regardless of any admissible realization of control v. The evasion problem consists in choosing the control v guaranteeing that system (1.1) evades contact with set M, regardless of any admissible realization of control u. Let us formulate the problem more precisely. Every triple $p = \{t; x; u(s), -\tau \leqslant s < 0\}$, where $t \in [t_0, \vartheta]$, $x \in E_n$ and $u(s) \equiv L^2[-\tau, 0)$, is called a position. Here $L^2[-\tau, 0)$ is the space of functions square summable on interval $[-\tau, 0)$. A rule associating a set $U(p) \subset P$ ($V(p) \subset Q$) with each game position p is called a strategy U (V). The initial position $p_0 = \{t_0; x_0; u_0(s), -\tau \leqslant s < 0\}$ is assumed given. Let Δ denote some covering of interval $[t_0, \vartheta]$ by the half-open intervals $\tau_i \leqslant t < \tau_{i+1}$, $\tau_0 = t_0, i = 0, 1, \ldots N(\Delta)$; let $\delta = \max_i (\tau_{i+1} - \tau_i)$. By $x[t, p_0, U]_\Delta$ we denote a function $x[t]_\Delta$, absolutely continuous on $[t_0, \vartheta]$, satisfying the intial condition

$$x[t_0]_\Delta = x_0 \tag{1.2}$$

and for almost all t from the interval $[t_0, \vartheta]$, the equation

$$x^{\cdot}[t]_\Delta = f_1(t, x[t]_\Delta, u[t], u[t-\tau]) + f_2(t, x[t]_\Delta, v[t]) \tag{1.3}$$

$$u.[t] = u[\tau_i] \in U(\tau_i; x[\tau_i]_\Delta; u_{\tau_i}[s], -\tau \leqslant s < 0)$$

$$t \in [\tau_i, \tau_{i+1}),\ i = 0, 1, \ldots, N$$
$$u[t - \tau] = u_0(t - \tau - t_0),\ t \in [t_0, t_0 + \tau);\ u_t(s) \equiv u(t + s),\ s \in [-\tau, 0)$$

Here $v[t]$ is some realization of the control, being an integrable time function with values in Q. Every continuous function possessing the following property: a sequence of coverings $\{\Delta_j\}$ with $\delta_j \to 0$ exists such that some sequence of functions $\{x[t, p_0, U]_{\Delta_j}\}$ converges uniformly on $[t_0, \vartheta]$ to $x[t, p_0, U]$, is called a motion $x[t] = x[t, p_0, U]$ from position p_0, corresponding to startegy U. A motion $x[t] = x[t, p_0, V]$ of system (1.1) from position p_0, corresponding to strategy V, is defined similarly.

P r o b l e m 1.1 (e n c o u n t e r). System (1.1), the time interval $[t_0, \vartheta]$, an initial position p_0, a closed bounded set $M \subset E_n$ and a number $c \geqslant 0$ are given. Construct the strategy U° guaranteeing the fulfilment of the condition $x[\vartheta] \in M^c$ for any motion $x[t] = x[t, p_0, U^\circ]$. Here M^c is the closed c-neighborhood of set M.

P r o b l e m 1. 2 (e v a s i o n). System (1.1), the time interval $[t_0, \vartheta]$, an initial position p_0, a closed bounded set $M \subset E_n$ and a number $c \geqslant 0$ are given. Construct the strategy V° guaranteeing the condition $x[\vartheta] \notin M^c$ for any motion $x[t] = x[t, p_0, V^\circ]$.

Sufficient solvability conditions for Problems 1.1. and 1.2 and a method for constructing the required control procedures are presented below.

2. Let a functional $\varepsilon(p) = \varepsilon(t;\ x;\ u_t(s),\ -\tau \leqslant s < 0)$ be specified on the space of positions, satisfying the following conditions:

1°. Functional $\varepsilon(p)$ is continuous under a change of position p, in the following sense: if the sequence of positions $\{p_k\} = \{\{t_k;\ x_k;\ u_{t_k}^{(k)}(s),\ -\tau \leqslant s < 0\}\}$ is such that $t_k \to t_*$ and $x_k \to x_*$ as $k \to \infty$ and $u^{(k)}(t_* + s) = u^*(t_* + s)$ when $s \in [-\tau, 0) \cap [t_k - t_* - \tau, t_k - t_*)$ for any k, then $\varepsilon(p_k) \to \varepsilon(p_*) = \varepsilon(t_*;\ x_*;\ u_{t_*}^*(s),\ -\tau \leqslant s < 0)$ as $k \to \infty$.

2° $\varepsilon(t;\ x;\ u_t^{(1)}(s),\ -\tau \leqslant s < 0) = \varepsilon(t;\ x;\ u_t^{(2)}(s),\ -\tau \leqslant s < 0)$ when $t \in [\vartheta - \tau, \vartheta]$ if only $u_t^{(1)}(s) = u_t^{(2)}(s)$ when $s \in [-\tau, \vartheta - \tau - t)$.

3° A number $c \geqslant 0$ exists such that $\varepsilon(\vartheta, x, u_\vartheta(s)) = \varepsilon(\vartheta, x) > c$ if $x \notin M^c$.

In addition, let the following conditions be fulfilled in the region $t < \vartheta$ and $c < \varepsilon(p) < \beta + c$, where $\beta > 0$.

4° The function $\varepsilon(t, x, u_t(s))$ possesses continuous partial derivatives $\partial\varepsilon / \partial x_i$, $i = 1, \ldots, n$, for fixed t and $u_t(s)$.

5° If function $u(t)$ is right-continuous at points t and $t - \tau$, then the representation

$$\varepsilon(t + \Delta t,\ x,\ u_{t+\Delta t}(s)) - \varepsilon(t,\ x,\ u_t(s)) = D(t,\ x,\ u_t(s),\ u(t),\ u(t - \tau))\,\Delta t + o(\Delta t) \tag{2.1}$$

is possible, where $\Delta t > 0$ and $D(t,\ x,\ u_t(s),\ u(t),\ u(t - \tau)) = D(p,\ u(t),\ u(t - \tau))$ is a functional continuous in all arguments, where the continuity with respect to a change in position p is understood in the sense of condition 1°.

6°. The inequality

$$\min_{u\in P}\left\{\frac{\partial\varepsilon}{\partial x} f_1(t, x, u, u(t-\tau))\right\} + D(t, x, u_t(s), u(t-\tau)) + \qquad (2.2)$$
$$\max_{v\in Q}\left\{\frac{\partial\varepsilon}{\partial x} f_2(t, x, v)\right\} \leqslant 0$$

is valid.

Note 2.1. In accord with condition 2° the functional $D(t, x, u_t(s), u(t), u(t-\tau))$ in (2.1) depends only on t, x, $u_t(s)$ and $u(t-\tau)$ when $t \in [\vartheta - \tau, \vartheta]$; therefore, condition 6° takes the form

$$\min_{u\in P}\left\{\frac{\partial\varepsilon}{\partial x} f_1(t, x, u, u(t-\tau)) + D(t, x, u_t(s), u, u(t-\tau))\right\} + \qquad (2.3)$$
$$\max_{v\in Q}\left\{\frac{\partial\varepsilon}{\partial x} f_2(t, x, v)\right\} \leqslant 0$$

Analogously to [2] we introduce the concept of an extremal strategy U°. If $\varepsilon(p) \leqslant c$ or $\varepsilon(p) \geqslant c + \beta$, we assume $U^\circ(p) = P$; if $c < \varepsilon(p) < c + \beta$, then $U^\circ(p)$ is the set of vectors $u^\circ \in P$ satisfying the condition

$$\min_{u\in P}\left\{\frac{\partial\varepsilon}{\partial x} f_1(t, x, u, u(t-\tau)) + \lambda D(t, x, u_t(s), u, u(t-\tau))\right\} =$$
$$\frac{\partial\varepsilon}{\partial x} f_1(t, x, u^\circ, u(t-\tau)) + \lambda D(t, x, u_t(s), u^\circ, u(t-\tau))$$
$$\lambda = \begin{cases} 0, & t \in [\vartheta - \tau, \vartheta] \\ 1, & t \in [t_0, \vartheta - \tau) \end{cases}$$

The following theorem is valid.

Theorem 2.1. Let a functional $\varepsilon(p)$ exist satisfying conditions 1° –3° and satisfying conditions 4° –6° in the region $t < \vartheta$ and $c < \varepsilon(p) < c + \beta$, where $\beta > 0$. Then, if $\varepsilon(p_0) \leqslant c$, the extremal startegy U° solves the encounter Problem 1.1.

The evasion problem is solved analogously.

Let a functional $\varepsilon(p)$ be given, satisfying conditions 1°, 2° and the following condition:

3°a. A number $c \geqslant 0$ exists such that $\varepsilon(\vartheta, x, u_\vartheta(s)) = \varepsilon(\vartheta, x) \leqslant c$ if $x \in M^c$.

In addition, let conditions 4°, 5° and the following condition 6°a be fulfilled in the region $t < \vartheta$ and $c - \gamma < \varepsilon(p) \leqslant c$, where $\gamma > 0$:

6°a. The inequality resulting from (2.2) when the sign $\leqslant$ is replaced by the sign $\geqslant$ is valid.

Note 2.2. When $t \in [\vartheta - \tau, \vartheta]$ the last inequality can be represented in form (2.3) with the sign $\leqslant$ replaced by the sign $\geqslant$.

We define the extremal strategy V° as follows: if $\varepsilon(p) > c$ or $\varepsilon(p) \leqslant c - \gamma$, then $V^\circ(p) = Q$; if $c - \gamma < \varepsilon(p) \leqslant c$, then $V^\circ(p)$ is the set of vectors $v^\circ \in Q$

satisfying the condition

$$\max_{v\in Q}\left\{\frac{\partial\varepsilon}{\partial x} f_2(t, x, v)\right\} = \frac{\partial\varepsilon}{\partial x} f_2(t, x, v^\circ)$$

The following theorem is valid.

Theorem 2.2. Let a functional $\varepsilon(p)$ exist satisfying conditions 1°, 2° and 3°a and satisfying conditions 4°, 5° and 6°a in the region $t < \vartheta$ and $c - \gamma < \varepsilon(p) \leqslant c$, where $\gamma > 0$. Then, if $\varepsilon(p_0) > c$, the extremal strategy V° solves the evasion Problem 1.2.

We obtain the solution of the differential game of encounter-evasion with target set M^c at instant ϑ by combining Theorems 2.1 and 2.2.

Theorem 2.3. Let a functional $\varepsilon(p)$ exist satisfying conditions 1° and 2°, the boundary conditions

$$\varepsilon(\vartheta, x) = \min_{m\in M}\{\| x - m \|\} \tag{2.4}$$

and conditions 4° $-$6° in some region $0 \leqslant \sigma_0 < \varepsilon(p) < \sigma^\circ$ and $t < \vartheta$, where (2.2) in condition 6° is fulfilled with the equality sign. Then for any initial position p_0 and for any number c such that $\sigma_0 < c < \sigma^\circ$ either a strategy U° exists such that $x[\vartheta] \in M^c$ is fulfilled for any motion $x[t] = x[t, p_0, U^\circ]$ or a strategy V° exists such that $x[\vartheta] \notin M^c$ is fulfilled for any motion $x[t] = x[t, p_0, V^\circ]$.

3. Let us discuss the possibility of constructing a functional ε with the properties required, relying on the results in [2]. We consider probability measures $\nu_t(dv)$ depending on $t \in [t_0, \vartheta)$ and defined on set Q, satisfying the condition of weak measurability: the function

$$\beta(t) = \int_Q \alpha(v)\,\nu_t(dv)$$

must be Lebesgue-measurable on $[t_0, \vartheta)$ for every continuous function $\alpha(v)$. We consider as well probability measures $\mu_t(du)$ depending on $t \in [t_0 - \tau, \vartheta)$ and defined on set P, satisfying an analogous condition of weak measurability. For every probability $\mu_t(du)$, weakly measurable on $[t_0 - \tau, \vartheta)$, we can define a measure $\mu_{t,t-\tau}(du, du^\tau) = \mu_{t-\tau}(du^\tau)\,\mu_t(du)$, weakly measurable on $[t_0, \vartheta)$, defined on set $P \times P$ for each value of $t \in [t_0, \vartheta)$. Having the function $\mu = \mu_t$, weakly measurable $[t_0 - \tau, \vartheta)$, and the function $\nu = \nu_t$, weakly measurable on $[t_0, \vartheta]$, we can construct measures on $[t_0, \vartheta) \times P \times P$ and $[t_0, \vartheta) \times Q$, respectively: $\mu^*(dt, du, du^\tau) = \mu_{t-\tau}(du^\tau)\,\mu_t(du)\,dt$ and $\nu^*(dt, dv) = \nu_t(dv)\,dt$. By the weak convergence of functions μ_t, $t \in [t_0, \vartheta)$ we mean weak convergence in the space of linear functions

$$\beta_{\mu^*}(\alpha) = \int_{t_0}^{\vartheta}\int_P\int_P \alpha(t, u, u^\tau)\,\mu_{t-\tau}(du^\tau)\,\mu_t(du)\,dt$$

defined on the space of functions $\alpha(t, u, u^\tau)$ defined and continuous on $[t_0, \vartheta) \times P \times P$. Correspondingly, the weak convergence of functions ν_t, $t \in [t_0, \vartheta)$, is weak convergence in the space of linear functionals

$$\beta_{\nu^*}(\alpha) = \int_{t_0}^{\vartheta}\int_Q \alpha(t, v)\, \nu_t(dv)\, dt$$

The sets of measures of form $\mu_{t-\tau}(du^\tau)\,\mu_t(du)\,dt$ and $\nu_t(dv)\,dt$, constructed above, are sets weakly closed and weakly compact in themselves. The weakly measurable functions $\mu = \mu_t$ $(\nu = \nu_t)$, $t \in [t_*, \vartheta)$, whose values are the probability measures $\mu_t(du)$ on P $(\nu_t(dv)$ on $Q)$ are called program controls on the half-open interval $[t_*, \vartheta)$. The weakly measurable functions $\mu = \mu_{t_*+s}$, $s \in [-\tau, 0)$, whose values are the probability measures $\mu_{t_*+s}(du)$ on P are called the prior histories of the program control to the instant t_*.

A solution of the differential equation with initial condition

$$x^\cdot = \int_P\int_P f_1(t, x, u, u^\tau)\,\mu_{t-\tau}(du^\tau)\,\mu_t(du) + \int_Q f_2(t, x, v)\,\nu_t(dv),$$

$$x(t_*) = x_*$$

is called a program motion $x(t, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t)$ generated by program controls μ_t and ν_t, by the prior history μ_{t_*+s}, $s \in [-\tau, 0)$, of the program control to the instant t_* and by the initial values t_* and x_*. We consider two auxiliary problems.

P r o b l e m 3.1. Given the triple $\{t_*;\ x_*;\ \mu_{t_*+s},\ -\tau \leqslant s < 0\}$, a bounded closed set $M \subset E_n$ and a program control ν_t, $t \in [t_*, \vartheta)$. Among the program controls μ_t find the optimal minimizing control μ_t°, $t \in [t_*, \vartheta)$, satisfying the condition

$$\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t^\circ, \nu_t), M) = \min_{\mu_t}\{\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t), M)\}$$

$$\rho(x, M) = \min_{m\in M}\{\| x - m \|\}$$

Problem 3.1 has a solution for every t_*, x_*, μ_{t_*+s} and ν_t. Indeed $\rho(x, M)$ depends continuously on x, while in its own turn $x = x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t)$ depends continuously on the program control μ_t, $t \in [t_*, \vartheta)$, as can be verified [2], if the proximity of the program controls μ_t to each other is estimated in the weak topology. Then the functional $\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t), M)$ achieves, on the weakly compact set $\{\mu_t,\ t \in [t_*, \vartheta)\}$ of its arguments, a minimum on some control μ_t°. The optimal program control μ_t° solving Problem 3.1 satisfies a certain condition that is an analog of Pontriagin's maximum principle transformed for systems with a nontrivial time lag in the control variable (see [6]).

T h e o r e m 3.1. Let the inequality

$$\min_{\mu_t}\{\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t), M)\} > 0$$

be fulfilled under the hypotheses of Problem 3.1. Then the optimal program control μ_t° and the program motion $x^\circ(t) = x(t, t_*, x_*, \mu_{t_*+s}, \mu_t^\circ, \nu_t)$ generated by it satisfy for almost all t from $[t_*, \vartheta]$ the conditions

$$\int_P\int_P s(t)\, f_1(t, x^\circ(t), u, u^\tau)\,\mathring{\mu}_{t-\tau}(du^\tau)\,\mu_t^\circ(du) +$$

$$\lambda \int_P \int_P s(t+\tau) f_1(t+\tau, x^\circ(t+\tau), u_\tau, u)\, \mu_t^\circ(du)\, \mathring{\mu}_{t+\tau}(du_\tau) =$$
$$\int_P \min_{u\in P} \{s(t) f_1(t, x^\circ(t), u, u^\tau)\, \mathring{\mu}_{t-\tau}(du^\tau) +$$
$$\lambda s(t+\tau) f_1(t+\tau, x^\circ(t+\tau), u_\tau, u)\, \mathring{\mu}_{t+\tau}(du_\tau)\}$$
$$\lambda = \begin{cases} 1, & t\in[t_*, \vartheta-\tau) \\ 0, & t\in[\vartheta-\tau, \vartheta] \end{cases}$$

Here $u_\tau = u(t+\tau)$ and $s(t)$ is a solution of the equation with boundary condition

$$s^{\cdot}(t) = -L(t)s(t), \quad s(\vartheta) = \frac{x^\circ(\vartheta) - m^\circ}{\|x^\circ(\vartheta) - m^\circ\|} \tag{3.1}$$
$$L(t) = \int_P\int_P \left[\frac{\partial f_1}{\partial x}\right]_{x^\circ(t)} \mathring{\mu}_{t-\tau}(du^\tau)\, \mu_t^\circ(du) + \int_Q \left[\frac{\partial f_2}{\partial x}\right]_{x^\circ(t)} \nu_t(dv)$$

where m° is the point of M closest to $x^\circ(\vartheta)$ (possibly, nonunique).

P r o b l e m 3.2. Given the triple $\{t_*, x_*; \mu_{t_*+s}, -\tau \leqslant s < 0\}$, a bounded closed set M and an instant ϑ. Among the program controls μ_t and ν_t, $t\in[t_*, \vartheta)$, find the optimal maximizing pair $\{\mu_t^\circ, \nu_t^\circ\}$ of controls, satisfying the condition

$$\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t^\circ, \nu_t^\circ), M) = \tag{3.2}$$
$$\min_{\mu_t} \{\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t^\circ), M\} =$$
$$\max_{\nu_t} \min_{\mu_t} \{\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t, \nu_t), M) = \varepsilon(t_*, x_*, \mu_{t_*+s})$$

By reasonings similar to the proof of existence of the solution of Problem 3.1 it can be verified that Problem 3.2 has a solution for every t_*, x_* and μ_{t_*+s}.

We say that regularity conditions are fulfilled in region $0 \leqslant \sigma_0 < \varepsilon < \sigma^\circ$ if for every triple $\{t_*; x_*; \mu_{t_*+s}, -\tau \leqslant s < 0\}$ such that $0 \leqslant \sigma_0 < \varepsilon(t_*, x_*, \mu_{t_*+s}) < \sigma^\circ$, Problem 3.2 has a unique solution $\{\mu_t^\circ, \nu_t^\circ\}$ (to within coincidence on a set of measure zero) and the value $m^{\circ\circ} \in M$ minimizing $\rho(x(\vartheta, t_*, x_*, \mu_{t_*+s}, \mu_t^\circ, \nu_t^\circ), M)$ is unique as well.

T h e o r e m 3.2. Let the regularity conditions be fulfilled in region $0 \leqslant \sigma_0 < \varepsilon < \sigma^\circ$. Then if $\sigma_0 < \varepsilon(t_*, x_*, \mu_{t_*+s}) < \sigma^\circ$, the optimal maximizing control ν_t° of Problem 3.2 satisfies the following condition:

$$\int_Q s(t) f_2(t, x^{\circ\circ}(t), v)\, \nu_t^\circ(dv) = \max_{v\in Q} \{s(t) f_2(t, x^{\circ\circ}(t), v)\}$$

for almost all $t\in[t_*, \vartheta]$. The conclusion of Theorem 3.1, wherein $x^\circ(t)$ should be replaced by $x^{\circ\circ}(t) = x(t, t_*, x(t, t_*, x_*, \mu_{t_*+s}, \mu_t^\circ, \nu_t^\circ)$ is fulfilled for the optimal minimizing control μ_t° of Problem 3.2. The value $s(t)$ is determined from (3.1) where m° should be replaced by $m^{\circ\circ}$, and $x^\circ(t)$ by $x^{\circ\circ}(t)$, and ν_t by ν_t°.

Theorems 3.1 and 3.2 are proved by proof plan for Lemma 36.1 and 37.1 in [2].

The quantity $\varepsilon(t_*, x_*, \mu_{t_*+s})$ is determined also when Lebesgue -measurable functions $u_{t_*}(s)$ mapping the half-open interval $[-\tau, 0)$ into P are prescribed instead of the functions μ_{t_*+s} whose values are the probability measures $\mu_{t_*+s}(du^\tau)$. Consequently, the quantity $\varepsilon(p)$ is defined for each position $p = \{t;\ x;\ u_t(s),\ -\tau \leqslant s < 0\}$. The sets of all $\{x,\ u_t(s)\}$ such that $\varepsilon(t,\ x,\ u_t(s)) \leqslant c$ are called program absorption sets $W_t{}^c$ of target M^c. Thus , $\{x_*,\ u_{t_*}(s)\} \in W_{t*}{}^c$ if and only if for every choice of program control $\nu_t\ (t \in [t_*,\ \vartheta))$, among the program controls $\mu_t\ (t \in [t_*,\ \vartheta))$ we can find at least one such that the inclusion $x(\vartheta) \in M^c$ is fulfilled for the program motion $x(t) = x(t,\ t_*,\ x_*,\ u_{t_*}(s),\ \mu_t,\ \nu_t)$.

Theorem 3.3. The functional $\varepsilon(p_*) = \varepsilon(t_*,\ x_*,\ u_{t_*}(s))$ defined by (3.2) satisfies conditions 1° and 2° and the boundary condition (2.4). If the regularity conditions are fulfilled in the region $0 \leqslant \sigma_0 < \varepsilon < \sigma^\circ$, then conditions 4° − 6° are fulfilled in this region, and

$$\left[\frac{\partial\varepsilon}{\partial x}\right]_{\{t_*,\, x_*,\, u_{t_*}(s)\}} = s(t_*) \tag{3.3}$$

$$\begin{aligned} &D(t_*, x_*, u_{t_*}(s), u(t_*), u(t_* - \tau)) = \\ &\quad \lambda \int_P s(t_* + \tau) f_1(t_* + \tau, x^{\circ\circ}(t_* + \tau), u_\tau, u(t_*))\, \mu^\circ_{t_*+\tau}(du_\tau) - \\ &\quad \min_{u\in P}\{s(t_*) f_1(t_*, x_*, u, u(t_* - \tau)) + \\ &\quad \lambda \int_P s(t_* + \tau) f_1(t_* + \tau, x^{\circ\circ}(t_* + \tau), u_\tau, u)\, \mu^\circ_{t_*+\tau}(du_\tau)\} - \\ &\quad \max_{v\in Q}\{s(t_*) f_2(t_*, x_*, v)\} \end{aligned} \tag{3.4}$$

$$\lambda = \begin{cases} 1, & t_* \in [t_0, \vartheta - \tau) \\ 0, & t_* \in [\vartheta - \tau, \vartheta] \end{cases}$$

and in condition 6° the bounds (2.2) and (2.3) are fulfilled with the equality sign. The quantities $\mu_t{}^\circ$, $\nu_t{}^\circ$, $x^{\circ\circ}(t)$ and $s(t)$ here are the same as in Theorem 3.2.

To prove Theorem 3.3 we can use the reasonings in the proofs of the analogous statements in [2].

4. As an example we consider the linear time-lag control system

$$x^\cdot(t) = A(t)\,x(t) + B_1(t)\,u(t) + B_2(t)\,u(t - \tau) - C(t)\,v(t) + w(t) \tag{4.1}$$

The matrices $A(t)$, $B_1(t)$, $B_2(t)$, $C(t)$ and $w(t)$ are continuous functions on $[t_0, \vartheta]$. We assume that sets P, Q and M are convex. To be specific we consider the Problem 1.1. of encounter with set M . The program controls here are any functions $u(t)$ ($v(t)$) Lebesgue −integrable on interval $[t_0, \vartheta]$, with values in P (Q) . Using the Cauchy formula and the separability condition for target set M and the attainability set, we can establish the form of the program absorption set W_t.

Theorem 4. 1. $\{x;\ u_t(s), -\tau \leqslant s < 0\} \in W_t$ if and only if $\gamma(t, x, u_t(s)) \leqslant 0$, where

$$\begin{aligned} \gamma(t,\ x,\ u_t(s)) = \max_{\|l\|=1}\Big\{ &\int_t^\vartheta \max_{v\in Q}\{lF(\vartheta, \xi)\, C(\xi)\, v(\xi)\}\, d\xi - \lambda I - \\ &\int_{\eta_1(\lambda)}^\vartheta \max_{u\in P}\{lF(\vartheta, \xi)\, B_1(\xi)\, u(\xi)\}\, d\xi - \int_t^\vartheta lF(\vartheta, \xi)\, w(\xi)\, d\xi - \end{aligned} \tag{4.2}$$

$$\int_{-\tau}^{\eta_2(\lambda)} lF(\vartheta, t+\tau+s) B_2(t+\tau+s) u_t(s)\,ds - lF(\vartheta, t)x + \min_{q\in M} lq\Big\},$$

$$l \in E_n, \quad \lambda = \begin{cases} 1, & t \in [t_0, \theta-\tau) \\ 0, & t \in [\theta-\tau, \theta] \end{cases}$$

$$I = \int_t^{\vartheta-\tau} \max_{u\in P} \{l[F(\vartheta, \xi) B_1(\xi) + F(\vartheta, \xi+\tau) B_2(\xi+\tau)]\, u(\xi)\}\, d\xi$$

$$\eta_1(\lambda) = \begin{cases} \vartheta-\tau, & \lambda = 1 \\ t, & \lambda = 0, \end{cases} \quad \eta_2(\lambda) = \begin{cases} 0, & \lambda = 1 \\ \vartheta-\tau-t, & \lambda = 0 \end{cases}$$

Here $F(\vartheta, \xi)$ is the fundamental matric for Eq. (4.1), i.e., an $n \times n$ -matrix with the properties $F(t, t) = E$ and $\partial F(t, \xi)/\partial t = A(t) F(t, \xi)$; E is the unit matrix.

We assume that the regularity conditions are fulfilled in region $0 < \varepsilon < \infty$. Using the results in Paragraphs 2 and 3 we construct the extremal strategy U°. The equation and the boundary condition for the quantity $s(t)$ in the linear case take the form

$$s^\cdot(t) = -A(t) s(t), \quad s(\vartheta) = -l^\circ$$

Then

$$s(t) = -F(\vartheta, t)\, l^\circ \tag{4.3}$$

where l° is the vector supplying the maximum in the expression for $\gamma(t, x, u_t(s))$ in (4.2). From (3.4), making appropriate changes, we obtain

$$\begin{aligned} D(t, x, u_t(s), u(t), u(t-\tau)) = \lambda s(t+\tau) B_2(t+\tau) u(t) - \min_{u\in P} \{s(t) B_1(t) + \\ \lambda s(t+\tau) B_2(t+\tau) u\} + \max_{v\in Q} \{s(t) C(t) v\} - s(t)\{A(t)x + B_2(t) u(t-\tau) + w(t)\} \end{aligned} \tag{4.4}$$

$$\lambda = \begin{cases} 1, & t \in [t_0, \vartheta-\tau) \\ 0, & t \in [\vartheta-\tau, \vartheta] \end{cases}$$

Using (4.3) and (4.4), we obtain by Theorems 3.3. and 2.1 that the extremal strategy U° solving in the regular case the problem of encounter with set M (if the initial position p_0 is such that $\gamma(p_0) \leqslant 0$) is specified as follows. If $\gamma(p) \leqslant 0$, then $U^\circ(p) = P$. If $\gamma(p) > 0$, then

$$\begin{aligned} l^\circ[F(\vartheta, t) B_1(t) + \lambda F(\vartheta, t+\tau) B_2(t+\tau)]\, u^\circ = \\ \max_{u\in P} \{l^\circ[F(\vartheta, t) B_1(t) + \lambda F(\vartheta, t+\tau) B_2(t+\tau)]\, u\} \end{aligned} \tag{4.5}$$

$$\lambda = \begin{cases} 1, & t \in [t_0, \vartheta-\tau) \\ 0, & t \in [\vartheta-\tau, \vartheta] \end{cases}$$

In the case given the regularity conditions signify, according to the definition in Paragraph 3 and to Theorems 3.1 and 3.2, that when $\gamma(p) > 0$, first, the vector l° supplying the maximum in the expression for $\gamma(p)$ is unique and, second, a unique (to within coincidence on a set of measure zero) control pair $\{u^\circ(t), v^\circ(t)\}$ exists, specified by conditions (4.5) and the condition

$$l^\circ F(\vartheta, t) C(t) v^\circ = \max_{v\in Q} \{l^\circ F(\vartheta, t) C(t) v\} \tag{4.6}$$

Note 4.1. In the example being analyzed we can weaken the regularity condition, requiring only the uniqueness of the vector l^0 supplying the maximum in the expression for $\gamma(p) > 0$ (see [1, 2]). This requirement reduces to the requirement that function $\chi_t(l)$ be concave in l, specified by the condition

$$\chi_t(l) = \int_t^{\vartheta} \max_{v\in Q}\{lF(\vartheta,\xi)C(\xi)v(\xi)\}\,d\xi - \lambda\int_t^{\vartheta-\tau}\max_{u\in P}\{l[F(\vartheta,\xi)B_1(\xi) +$$

$$F(\vartheta,\xi+\tau)B_2(\xi+\tau)]u(\xi)\}\,d\xi - \int_{\eta_1}^{\vartheta}\max_{u\in P}\{lF(\vartheta,\xi)B_1(\xi)u(\xi)\}d\xi + \min_{q\in M} lq$$

$$\lambda = 1,\quad \eta_1 = \vartheta - \tau,\quad t \in [t_0,\ \vartheta - \tau)$$

$$\lambda = 0,\quad \eta_1 = t,\quad t \in [\vartheta - \tau,\ \vartheta]$$

Note 4.2. The function $\chi_t(l)$ is certainly concave in l for any $t \in [t_0, \vartheta]$ if a convex set $R(t)$ exists such that

$$[F(\vartheta,t)B_1(t) + \lambda F(\vartheta, t+\tau)B_2(t+\tau)]P = F(\vartheta,t)C(t)Q + R(t)$$

$$\lambda = \begin{cases} 1, & t\in[t_0,\ \vartheta-\tau) \\ 0, & t\in[\vartheta-\tau,\ \vartheta] \end{cases}$$

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Translated by N.H.C.